CG2023 Finals Cheatsheet by Luke Aidan Tan

Energy & Power

Total energy of $x(t)$:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \,\mathrm{d}t$$

$x(t)$ is an **energy signal** iff $0 < E < \infty$.
Total power of $x(t)$:

$$P = \lim_{\tau \to \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} |x(t)|^2 \,\mathrm{d}t$$

$x(t)$ is a **power signal** iff $0 < P < \infty$.
Remarks. Energy signals have zero avg power. Power signals have infinite total energy.

Signal Transformations

$x(at+b)$: scale time by $\frac{1}{|a|}$, shift by $-\frac{b}{a}$.
Energy/power scaling: $ax\left(\frac{t}{b}\right)$ gives $E' = a^2 bE$, $P' = a^2 P$.

Convolution & Unit Impulse

Convolution

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\alpha) y(t-\alpha) \,\mathrm{d}\alpha$$

Properties: Commutative, Associative, Distributive.

Dirac-δ Function

$\delta(t) = 0$ for $t \neq 0$; $\int_{-\varepsilon}^{\varepsilon} \delta(t) \,\mathrm{d}t = 1$.

Properties:

- Symmetry: $\delta(t) = \delta(-t)$
- Sampling: $x(t)\delta(t-\lambda) = x(\lambda)\delta(t-\lambda)$
- Sifting: $\int_{-\infty}^{\infty} x(t)\delta(t-\lambda) \,\mathrm{d}t = x(\lambda)$
- Replication: $x(t) * \delta(t-\lambda) = x(t-\lambda)$

Dirac-δ Comb

$$\sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Generating function: $x(t) * \sum_n \delta(t-kn) = \sum_n x(t-kn)$

Fourier Series

Any bounded periodic signal $x_{p(t)}$ with period T_p:

$$x_p(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_p t}, \quad f_p = \frac{1}{T_p}$$

Analysis (coefficients):

$$c_k = \frac{1}{T_p} \int_{t_0}^{t_0+T_p} x_{p(t)} e^{-j2\pi k \frac{t}{T_p}} \,\mathrm{d}t$$

Trigonometric form:

$$x_p(t) = a_0 + 2\sum_{k=1}^{\infty} \left[a_k \cos\left(2\pi k \frac{t}{T_p}\right) + b_k \sin\left(2\pi k \frac{t}{T_p}\right) \right]$$

where $a_k = \frac{c_{-k}+c_k}{2}$, $b_k = \frac{c_{-k}-c_k}{j2}$.

Spectral Symmetry (Fourier Series)

Signal type	Coefficient properties
Real	$c_k^* = c_{-k}$ $\lvert c_k \rvert = \lvert c_{-k} \rvert$ $\angle c_k = -\angle c_{-k}$
Real & even	$c_k^* = c_k$ (real) $c_k = c_{-k}$ (even)
Real & odd	$c_k^* = -c_k$ (imaginary) $-c_k = c_{-k}$ (odd)

Fourier Transform

Dirichlet Conditions

1. Finite maxima/minima in any finite interval.
2. Finite discontinuities in any finite interval.
3. Absolutely integrable: $\int_{-\infty}^{\infty} |x(t)| \,\mathrm{d}t < \infty$.

Condition 3 is violated by power signals; (1) and (2) are satisfied by all real-world signals.

Spectral Symmetry (FT)

Signal	Spectrum properties
Real $x^*(t) = x(t)$	$X^*(f) = X(-f)$ (conj. symm.) $\lvert X(f) \rvert = \lvert X(-f) \rvert$ (even) $\angle X(f) = -\angle X(-f)$ (odd)
Real & even $x^*(t) = x(t)$, $x(-t) = x(t)$	$X^*(f) = X(f)$ (real) $X(f) = X(-f)$ (even)
Real & odd $x^*(t) = x(t)$, $x(-t) = -x(t)$	$X^*(f) = -X(f)$ (imaginary) $-X(f) = X(-f)$ (odd)

Periodic Signals via FT

If $x_p(t)$ has Fourier series coefficients c_k:

$$X_p(f) = \sum_{k=-\infty}^{\infty} c_k \delta(f - kf_p)$$

Dirac-δ comb spectrum: $\mathrm{COMB}_\lambda(f) = \frac{1}{\lambda} \sum_k \delta\left(f - \frac{k}{\lambda}\right)$

Spectral Density & Bandwidth

Energy Spectral Density

Parseval/Rayleigh: $E = \int |x(t)|^2 \,\mathrm{d}t = \int |X(f)|^2 \,\mathrm{d}f$

$$E_x(f) = |X(f)|^2$$

Properties: real, ≥ 0, even if $x(t)$ real.

Power Spectral Density

$$P_x(f) = \lim_{W\to\infty} \frac{1}{2W} |X_W(f)|^2, \quad P = \int P_x(f) \,\mathrm{d}f$$

For periodic $x_p(t)$: $P_x(f) = \sum_k |c_k|^2 \, \delta(f - kf_p)$, $P = \sum_k |c_k|^2$

Bandwidth Definitions

Lowpass signal: $|X(f)| = 0$ for $|f| > B$.
Bandpass signal: $|X(f)| = 0$ for $\||f| - f_c| > \frac{B}{2}$.
3 dB BW (LP): smallest f_B where $|X(f_B)| = |X(0)|/2^{0.5}$ (-3.01 dB).
3 dB BW (BP): smallest f_B where $|X(f_B)| = |X(f_c)|/2^{0.5}$; $B = f_u - f_l$.
1st-null BW (LP): smallest f where $|X(f)| = 0$.
1st-null BW (BP): smallest $f_u - f_l$ around f_c where $|X(f)| = 0$.
M% Energy BW (LP): smallest B s.t. $\int_{-B}^{B} E_x(f) \,\mathrm{d}f \geq \left(\frac{M}{100}\right) E$
M% Energy BW (BP): smallest B s.t. $\int_{f_c-\frac{B}{2}}^{f_c+\frac{B}{2}} E_x(f) \,\mathrm{d}f \geq \left(\frac{M}{100}\right)\left(\frac{E}{2}\right)$
M% Power BW (periodic): $B = Kf_p$ where K is smallest integer s.t.

$$\sum_{k=-K}^{K} |c_k|^2 \geq \frac{M}{100} \times P$$

Systems — Classification

System Types

Property	Condition
Memoryless	Output depends only on input at same t
Causal	Output depends only on present/past input
BIBO Stable	Every bounded input $\Rightarrow$ bounded output
Linear	Superposition holds (additivity + homogeneity)
Time-Invariant	$x(t-\tau) \Rightarrow y(t-\tau)$ for all τ

Linearity (Superposition)

$$\alpha_1 x_1(t) + \alpha_2 x_2(t) \to \alpha_1 y_1(t) + \alpha_2 y_2(t)$$

Zero input $\Rightarrow$ zero output (set all $\alpha_i = 0$).

Laplace Transform (Definition)

$$F(s) = \int_0^{\infty} f(t) e^{-st} \,\mathrm{d}t, \quad s = \sigma + j\omega$$

Partial Fractions — Method

Step 1. If $\deg(N) \geq \deg(D)$, perform polynomial long division first.
Simple real roots $F(s) = \frac{N(s)}{(s+a)(s+b)}$:

$$F(s) = \frac{A}{s+a} + \frac{B}{s+b}, \quad A = [(s+a)F(s)]_{s=-a}$$

Repeated root $(s+a)^r$:

$$\ldots + \frac{A_1}{s+a} + \frac{A_2}{(s+a)^2} + \ldots + \frac{A_r}{(s+a)^r}$$

$$A_k = \frac{1}{(r-k)!} \left[\frac{\mathrm{d}^{r-k}}{\mathrm{d}} s^{r-k} (s+a)^r F(s) \right]_{s=-a}$$

Complex conjugate roots $s^2 + 2\alpha s + \omega_n^2$: write as

$$\ldots + \frac{As+B}{s^2+2\alpha s+\omega_n^2}$$

Complete the square: $(s+\alpha)^2 + \omega_d^2$, then match to damped sin/cos pairs.
Solving DEs: Apply $\mathcal{L} \to$ rearrange for $Y(s) \to$ partial fractions $\to$ apply $\mathcal{L}^{-1}$.

LTI Systems

Representations

Domain	Key function
Time	Impulse response $h(t)$
Frequency	Freq. response $H(f)$ or $H(j\omega)$
s-domain	Transfer function $H(s)$

Convolution (Output)

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) \,\mathrm{d}\tau$$

Transfer Function

$$H(s) = \mathcal{L}\{h(t)\} = \frac{Y(s)}{X(s)}$$

(assumes zero initial conditions)

Frequency Response

$$H(f) = \mathcal{F}\{h(t)\}, \quad H(j\omega) = H(s)|_{s=j\omega}$$

For causal LTI: $H(f) = H(j\omega)|_{\omega=2\pi f}$.
Polar form: $H(j\omega) = |H(j\omega)| \, e^{j\angle H(j\omega)}$

Sinusoidal Steady-State

Input $A\cos(\omega_0 t + \psi) \Rightarrow$ output:

$$y(t) = A|H(j\omega_0)| \cos(\omega_0 t + \psi + \angle H(j\omega_0))$$

Amplitude scaled by $|H(j\omega_0)|$; phase shifted by $\angle H(j\omega_0)$. Same rule applies for sin.

General LTI Differential Equation

$$\sum_{n=0}^{N} a_n \frac{\mathrm{d}^n y}{\mathrm{d}t^n} = \sum_{m=0}^{M} b_m \frac{\mathrm{d}^m x}{\mathrm{d}t^m}$$

$$H(s) = \frac{b_M s^M + \ldots + b_0}{a_N s^N + \ldots + a_0}$$

Poles $-p_n$: roots of denominator; $H(-p_n) = \infty$.
Zeros $-z_m$: roots of numerator; $H(-z_m) = 0$.
Pole-zero excess $= N - M$.

System Stability (from Poles)

Type	Pole location	$h(t)$ as $t \to \infty$
BIBO Stable	All in LHP ($\mathrm{Re}[s] < 0$)	$\to 0$
Marginally stable	Simple poles on $j\omega$-axis	Bounded $\neq 0$
Unstable (I)	Any pole in RHP ($\mathrm{Re}[s] > 0$)	$\to \infty$
Unstable (II)	Repeated poles on $j\omega$-axis	$\to \infty$

Zeros do NOT affect stability — poles only.

Step Response

$$o(t) = \int_{-\infty}^{t} h(\tau) \,\mathrm{d}\tau = \mathcal{L}^{-1}\left\{\frac{H(s)}{s}\right\}$$

First-Order System

$$T\frac{\mathrm{d}y}{\mathrm{d}t} + y(t) = Kx(t) \quad \rightarrow \quad H(s) = \frac{K}{Ts+1}$$

- Pole: $s = -\frac{1}{T}$; Time constant: T
- Impulse resp.: $h(t) = \left(\frac{K}{T}\right)e^{-\frac{t}{T}}u(t)$
- Step resp.: $o(t) = K\left(1 - e^{-\frac{t}{T}}\right)u(t)$; reaches 63.2% of K at $t = T$

Second-Order System

$$H(s) = \frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$\text{Poles: } s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

ζ	Type	Poles
> 1	Overdamped	Real, distinct
$= 1$	Critically damped	Real, repeated: $-\omega_n$
$0 < \zeta < 1$	Underdamped	Complex conjugate
$= 0$	Undamped	Pure imaginary $\pm j\omega_n$

Underdamped ($0 < \zeta < 1$): $\sigma = \zeta\omega_n$, $\omega_d = \omega_n\sqrt{1-\zeta^2}$

Impulse resp.: $h(t) = \left(K\frac{\omega_n^2}{\omega_d}\right)e^{-\sigma t}\sin(\omega_d t)u(t)$

Step resp.: $o(t) = K\left[1 - e^{-\sigma t}\left(\cos\omega_d t + \left(\frac{\sigma}{\omega_d}\right)\sin\omega_d t\right)\right]u(t)$

Critically damped ($\zeta = 1$):

Impulse resp.: $h(t) = K\omega_n^2 t e^{-\omega_n t}u(t)$

Step resp.: $o(t) = K[1 - e^{-\omega_n t}(1 + \omega_n t)]u(t)$

Identifying parameters from $H(s) = \frac{K\omega_n^2}{s^2+b_1 s+b_0}$:

- $\omega_n = \sqrt{b_0}$
- $\zeta = \frac{b_1}{2\omega_n}$
- DC gain $= K$

Bode Diagrams

Bode plot = straight-line approx. of $H(j\omega)$ on semilog-ω axes.

- **Magnitude:** $|H(j\omega)|_{\text{dB}} = 20\log_{10}|H(j\omega)|$
- **Phase:** $\angle H(j\omega)$ in degrees

Standard Form

$$H(s) = K_{\text{dc}} \cdot \frac{\prod_m\left(1+\frac{s}{z_m}\right)}{\prod_n\left(1+\frac{s}{p_n}\right)} \cdot \prod_q \frac{\omega_{n,q}^2}{s^2+2\zeta_q\omega_{n,q}s+\omega_{n,q}^2}$$

where $K_{\text{dc}} = H(0)$ (DC gain).

With a differentiator: $K_d \cdot s \cdot \frac{\dots}{\dots}$; integrator: $\left(\frac{K_i}{s}\right) \cdot \frac{\dots}{\dots}$.

Building Blocks

Block	Mag slope	Phase
DC gain K_{dc}	0 dB/dec	0°
Differentiator $K_d s$	+20 dB/dec	+90°
Integrator $\frac{K_i}{s}$	−20 dB/dec	−90°
Zero $\left(1+\frac{s}{z_m}\right)$	+20 after z_m	0° → +90°
Pole $\frac{1}{1+\frac{s}{p_n}}$	−20 after p_n	0° → −90°
2nd-order factor	−40 after ω_n	0° → −180°

Phase transition spans one decade either side of the corner frequency (±45°/dec).

Asymptotic Rules

N = no. of poles, M = no. of zeros.

HF phase: $(N - M) \times -90°$

LF phase: $\left(\text{No. of } \int \mathrm{d}t - \text{No. of } \frac{\mathrm{d}}{\mathrm{d}t}\right) \times -90°$

HF slope: $(N - M) \times -20$ dB

LF slope: $\left(\text{No. of } \int \mathrm{d}t - \text{No. of } \frac{\mathrm{d}}{\mathrm{d}t}\right) \times -20$ dB

Reading Transfer Function from Bode Plot

1. **Low-freq slope:** $+20L$ dB/dec ⇒ L differentiators; $-20L$ ⇒ L integrators; 0 ⇒ DC gain K_{dc}.
2. Read DC gain from low-freq intercept.
3. Each **slope change of +20** dB/dec ⇒ zero at that corner freq.
4. Each **slope change of −20** dB/dec ⇒ pole.
5. **Slope change of −40** dB/dec ⇒ 2nd-order factor (check for resonance hump).
6. Assemble into $H(s) = \frac{K\left(1+\frac{s}{z_1}\right)}{\left(1+\frac{s}{p_1}\right)\cdot\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}}$ form.

Resonance (2nd-Order)

Valid only for $\zeta < \frac{1}{\sqrt{2}} \approx 0.707$ (i.e. $\zeta^2 < \frac{1}{2}$):

$$\omega_r = \omega_n\sqrt{1-2\zeta^2}, \quad |H(j\omega_r)|_{\text{peak}} = \frac{K}{2\zeta\sqrt{1-\zeta^2}}$$

When $\zeta \to 0$: $\omega_r \to \omega_n$, peak $\to \infty$.

Sampling & Reconstruction

Ideal Filters

Ideal LPF (cutoff B Hz, gain A):

$$H(f) = A \cdot \text{rect}\left(\frac{f}{2B}\right), \quad h(t) = 2AB\ \text{sinc}(2Bt)$$

Ideal BPF (centre f_o, BW $B = f_u - f_l$):

$$H(f) = A\left[\text{rect}\left(\frac{f-f_o}{B}\right) + \text{rect}\left(\frac{f+f_o}{B}\right)\right], \quad h(t) = 2AB\ \text{sinc}(Bt)\cos(2\pi f_o t)$$

Both are **non-causal** and physically unrealizable; approximated by Butterworth, Chebyshev, etc.

Sampling Process

Multiply $x(t)$ by impulse train with period $T_s = \frac{1}{f_s}$:

$$x_s(t) = x(t) \cdot \sum_n \delta(t - nT_s)$$

Spectrum (replicated at multiples of f_s):

$$X_s(f) = f_s \sum_{k=-\infty}^{\infty} X(f - kf_s)$$

Nyquist Sampling Theorem

For a signal band-limited to f_m Hz:

$$f_s \geq 2f_m \quad (\text{Nyquist rate} = 2f_m)$$

- $f_s \geq 2f_m$: **no aliasing**, perfect reconstruction possible.
- $f_s < 2f_m$: **aliasing** — spectral images overlap, reconstruction impossible.

Reconstruction Filter

Use ideal LPF with:

$$K = \frac{1}{f_s} = T_s, \quad f_m \leq f_c \leq f_s - f_m$$

Best choice: $f_c = f_m$ (minimum bandwidth). Output: $\hat{x}(t) = x(t)$.

Bandpass Sampling (below Nyquist rate)

Bandpass signal: centre f_c, bandwidth B. Full Nyquist rate $= 2\left(f_c + \frac{B}{2}\right)$.

Overlapping images — condition 9.2a (Symmetric):

$$f_s = \frac{2f_c}{k}, \quad k = 1, 2, ..., \left\lfloor\frac{2f_c}{B}\right\rfloor, \quad f_{s,\min} = \frac{2f_c}{\lfloor 2f_c/B \rfloor}$$

$$H(f) = \frac{1}{2f_{s,\min}}\left[\text{rect}\left(\frac{f+f_c}{B}\right) + \text{rect}\left(\frac{f-f_c}{B}\right)\right]$$

Un-aliased images — condition 9.2b (Asymmetric):

$$f_s \in \left[\frac{2f_c+B}{k+1}, \frac{2f_c-B}{k}\right], \quad k = 1, 2, ..., \left\lfloor\frac{2f_c-B}{2B}\right\rfloor, \quad f_{s,\min} = \frac{2f_c+B}{\lfloor (2f_c-B)/(2B) \rfloor + 1}$$

$$H(f) = \frac{1}{f_{s,\min}}\left[\text{rect}\left(\frac{f+f_c}{B}\right) + \text{rect}\left(\frac{f-f_c}{B}\right)\right]$$

Minimum sub-Nyquist rate $\approx 2B$ (equal to signal bandwidth).

Reconstruction filter: Ideal BPF covering the signal band (not LPF).

Quick Solve Methods

Time Scaling/Translation Hack by YTT

Expressing f(t) in terms of h(t),

$f(t) = h(at + b)$

Solving DEs (Laplace Method)

1. Apply $\mathcal{L}$ with zero ICs: $\frac{\mathrm{d}^n y}{\mathrm{d}t^n} \leftrightarrow s^n Y(s)$.
2. Rearrange: $Y(s) = H(s) \cdot X(s)$.
3. Expand via partial fractions.
4. Apply $\mathcal{L}^{-1}$ using table.

Finding Steady-State Output

1. Identify ω_0 from input.
2. Compute $H(j\omega_0)$: find magnitude and phase.
3. Scale and shift: $y(t) = A|H(j\omega_0)|\cos(\omega_0 t + \psi + \angle H(j\omega_0))$.

Stability Check

- Compute poles (roots of denominator of $H(s)$).
- All poles in LHP ⇒ BIBO stable.
- Any pole in RHP ⇒ unstable.
- Simple poles on $j\omega$-axis ⇒ marginally stable.
- Repeated poles on $j\omega$-axis ⇒ unstable.

Bode Plot (Step-by-Step)

1. Rewrite $H(s)$ in standard form (factored, unity-coefficient building blocks).
2. Compute K_{dc} (or K_d/K_i) and starting level in dB.
3. List corner frequencies in ascending order.
4. Draw low-freq asymptote; add slope changes at each corner.
5. Phase: each pole/zero contributes ±45°/dec over one decade either side of its corner.
6. Sum all contributions (dB magnitudes add linearly).

Complex Number Polar Form

$$a + jb = \sqrt{a^2+b^2} \cdot e^{j\tan^{-1}\left(\frac{b}{a}\right)}$$

$$\frac{1}{c+jd} = \frac{1}{\sqrt{c^2+d^2}} \cdot e^{-j\tan^{-1}\left(\frac{d}{c}\right)}$$

Angle Correction Quick-Reference

For $H(j\omega) = \frac{a+jb}{\dots}$, after computing $\theta = \arctan\left(\frac{b}{a}\right)$:

- $a > 0$: use θ as-is
- $a < 0, b > 0$: add 180° (2nd quadrant)
- $a < 0, b < 0$: subtract 180° (3rd quadrant)

Euler Identities

$$e^{j0} = 1, \quad e^{j\frac{\pi}{2}} = j, \quad e^{-j\frac{\pi}{2}} = -j, \quad e^{\pm j\pi} = -1, \quad e^{j\frac{3\pi}{2}} = -j$$

Roots of a Quadratic

For $ax^2 + bx + c = 0$, the roots are:

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Discriminant ($D = b^2 - 4ac$)

- $D > 0$: 2 real distinct roots.
- $D = 0$: 2 real repeated roots (Critically damped).
- $D < 0$: 2 complex conjugate roots (Underdamped).

Laplace Differentiation

$$\mathcal{L}\{x'(t)\} = sX(s) - x(0)$$

$$\mathcal{L}\{x''(t)\} = s^2X(s) - sx(0) - x'(0)$$